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HIGH-ORDER-ACCURATE SCHEMES FOR INCOMPRESSIBLE VISCOUS FLOW

JOHN C. STRIKWERDA

Computer Sciences Department, University of Wisconsin-Madison, Madison, WI 53706, U.S.A.

SUMMARY

We present new finite difference schemes for the incompressible Navier–Stokes equations. The schemes are based on two spatial differencing methods; one is fourth-order-accurate and the other is sixth-order accurate. The temporal differencing is based on backward differencing formulae. The schemes use non-staggered grids and satisfy regularity estimates, guaranteeing smoothness of the solutions. The schemes are computationally efficient. Computational results demonstrating the accuracy are presented. © 1997 by John Wiley & Sons, Ltd.

KEY WORDS: incompressible Navier–Stokes; finite difference schemes; GMRES

1. INTRODUCTION

High-order-accurate finite difference schemes are important in scientific computation because they offer a means to obtain accurate solutions with less work than may be required for methods of lower accuracy. Finite difference methods are attractive because of the relative ease of implementation and flexibility.

In this paper we present new finite difference schemes for the incompressible Navier–Stokes equations. The schemes are based on two spatial differencing methods, one a fourth-order-accurate method and one a sixth-order-accurate method. There are several temporal differencing methods presented in Section 7. These temporal schemes can be used with either of the spatial differencing methods. The temporal differencing is based on backward differencing formula (BDF) schemes that are used for stiff ordinary differential equations. The schemes are implicit and appear to be unconditionally stable for the Stokes equations. (A rigorous stability analysis is the subject of further research.)

High-order methods have been presented by Rai and Moin[1] and Lele[2] for the fractional step method proposed by Kim and Moin,[3] There is an excellent study of these methods in the paper by Tafti.[4] A disadvantage of these methods is that because they are explicit, there is a severe stability limit on the time step. Moreover, as pointed out by Perot,[5] the pressure for fractional step methods can be no better than first-order-accurate in time. Projection methods also have difficulty with higher-order accuracy in time.[6] This is not so for the methods presented here, where the pressure can be determined to a high order of accuracy. For steady flows the method of Aubert and Deville[7] can be applied to yield fourth-order accuracy, at the expense of increasing the number of unknowns and computational complexity of the system. All these methods use staggered meshes.

The schemes presented in this paper are for orthogonal Cartesian grids on non-staggered grids, the velocity components and pressure unknowns are assigned to a common grid. The schemes are for the

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two-dimensional Navier–Stokes equations; however, the methods for obtaining the equations extend easily to three dimensions and to generalizations of the Navier–Stokes equations. A second-order scheme similar to the ones presented here was presented in Reference 8, although it used a less efficient solution procedure. The methods presented here have been incorporated into a domain decomposition method.

The schemes presented here have been tested on sample problems with low Reynolds numbers. The tests show that the schemes are very accurate and efficient for these low-Reynolds-number computations. The extension of these methods to high Reynolds number is the subject of further research.

The non-dimensional time-dependent incompressible Navier–Stokes system of equations is

$$\vec{u}_t - \frac{1}{R}\nabla^2\vec{u} + \vec{\nabla}(\vec{u}\vec{u}^{\mathrm{T}}) + \vec{\nabla}p = \vec{f}, \tag{1}$$

$$\vec{\nabla}\cdot\vec{u} = g. \tag{2}$$

The vector function $\vec{u}$ is the velocity and the scalar function p is the pressure. The Reynolds number R measures the strength of the inertial effects relative to the viscous effects. Notice that the pressure appears only in (1) and only in terms of its spatial derivatives. We refer to equations (1) as the momentum equations and equation (2) as the divergence equation.

The functions $\vec{f}$ and g are considered to be given data. In most problems the function g in (2) is identically zero, but we include the general case because it fits in naturally with our methods and is useful in checking the accuracy of the computer implementation of the methods. In particular, the accuracy can be checked by choosing the velocity and pressure to be arbitrary polynomials of the proper degree (see Section 8).

In the limit as R tends to zero, with a rescaling of t, p and $\vec{f}$ the Navier–Stokes system can be replaced by the time-dependent Stokes system

$$\vec{u}_t - \nabla^2\vec{u} + \vec{\nabla}p = \vec{f}, \tag{3}$$

$$\vec{\nabla}\cdot\vec{u} = g. \tag{4}$$

We consider the Navier–Stokes sytem holding in a domain Ω; to specify a unique solution, boundary conditions must be given. The simplest conditions are to specify the velocity $\vec{u}$ on the boundary, i.e.

$$\vec{u} = \vec{b} \quad \text{on } \partial\Omega. \tag{5}$$

This is called the Dirichlet boundary condition. To limit our discussion, we only consider Dirichlet boundary conditions in this paper. The modifications needed for other boundary conditions should not be difficult to implement.

The system (1), (2) has a solution only if the integrability condition

$$\int_{\partial\Omega}\vec{n}\cdot\vec{b} = \int_{\Omega} g \tag{6}$$

is satisfied. This condition is a constraint relating the function g in (2) and (4) and the boundary data $\vec{b}$ in (5).

The schemes we develop are derived using the difference calculus. By considering the total system in the derivation, we obtain schemes that are compact, i.e. the stencil of the scheme is about as small as possible. In particular, the schemes presented here have smaller stencils than those of Lele[2] and Rai and Moin.[1] However, to obtain usable schemes, two other aspects must be taken into account. These are the regularity of the scheme and the behaviour at boundaries. The regularity of the scheme is important to assure that the solutions are smooth, i.e. the high-frequency modes are prevented from dominating the error. The schemes have parameters that remove these high-frequency modes, often referred to as checker-board pressure oscillations. The difficulty at the boundaries is related to the size of the stencil. Since the stencil increases in size as the order increases, the amount of modification required at the boundary also increases. These topics are addressed in Sections 5 and 6.

The schemes we derived can be used for both the steady state and time-dependent equations. We consider only schemes for which the temporal differencing and spatial differencing are independent of each other. The spatial differencing is discussed in Sections 3 and 4 and the temporal differencing in Section 7.

We avoid modifying the Navier–Stokes equations such as is done with the 'Poisson pressure equation' method. One difficulty with such methods is the need to decide on additional boundary conditions, especially on the pressure. This is also true for projection methods.[9] In our approach the linear systems that must be solved to determine the solution at each time step involve both the momentum and divergence equations. These large systems are solved by preconditioned GMRES methods.[10] One advantage of our approach is that the pressure can be obtained with the same order of accuracy in space as the velocity and better than first-order in time, which is the limit with fractional step methods.[5]

We do not use the finite volume approach, relying on the power of the symbolic difference calculus to obtain high accuracy with compact stencils. Our schemes do not satisfy exact conservation laws for mass or momentum. The accuracy of the solutions implies that the conservation laws should be satisfied to a high degree of accuracy. The schemes are based on the conservation form of the differential equations.

The structure of the paper is as follows. In Section 2 we present the notation for the basic difference operators. In Sections 3 and 4 we present the spatial differencing methods for orders four and six respectively. In section 5 we discuss the numerical boundary conditions needed for both schemes. Section 6 discusses the regularity of the two schemes for steady state computations. The multistep schemes used for the temporal differencing are discussed in Section 7. In Section 8 the numerical tests of the methods are discussed. Conclusions are presented in Section 9.

2. NOTATION

We develop our schemes for regular two-dimensional Cartesian grids with grid spacing Δx and Δy respectively. We use the notation δ_{x0} to denote the first-order central difference with respect to x, which is defined by

$$\delta_{x0} f_i = \frac{f_{i+1} - f_{i-1}}{2\Delta x}.$$

The forward and backward operators are denoted δ_{x+} and δ_{x-} respectively and are defined by

$$\delta_{x+} f_i = \frac{f_{i+1} - f_i}{\Delta x}, \qquad \delta_{x-} f_i = \frac{f_i - f_{i-1}}{\Delta x}.$$

The standard second-order central difference is denoted δ_x^2 and is defined by

$$\delta_x^2 f_i = \delta_{x+}\delta_{x-} f_i = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2}. \tag{7}$$

Difference operators δ_{y0}, δ_{y+}, etc. are defined similarly.

We obtain most of our difference formulae from basic identities relating derivatives to the difference operators δ_x^2 and δ_{x0}. For the first derivative we use the identity[11]

$$\frac{\partial}{\partial x} = \left[1 + \left(\frac{\Delta x \delta_x}{2}\right)^2\right]^{-1/2} \frac{\sinh^{-1}(\frac{1}{2}\Delta x \delta_x)}{\frac{1}{2}\Delta x \delta_x} \delta_{x0}. \tag{8}$$

Note that the expansion of this expression in terms of $\Delta x \delta_x$ contains only even powers; thus we need not explicitly define δ_x, needing only (7) to define δ_x^2.

The basic identity we use that relates the second derivative to δ_x^2 is[11]

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\sinh^{-1}(\frac{1}{2}\Delta x \delta_x)}{\frac{1}{2}\Delta x}\right)^2. \tag{9}$$

By expanding these expressions as Taylor series in Δx to appropriate powers, we may obtain difference approximations of any order.

To handle the modifications at the boundaries, we use two formulae that relate forward and backward differences:

$$\delta_{x-} = \frac{\delta_{x+}}{1 + \Delta x \delta_{x+}}, \qquad \delta_{x+} = \frac{\delta_{x-}}{1 - \Delta x \delta_{x-}}. \tag{10}$$

These two relations both arise from the identity

$$\delta_x^2 = \delta_{x+}\delta_{x-} = \frac{\delta_{x+} - \delta_{x-}}{\Delta x}.$$

3. THE FOURTH-ORDER SCHEME

Our fourth-order scheme for the Navier–Stokes equations is based on the approximations

$$\frac{\partial}{\partial x} = \left(1 - \frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0} + O(\Delta x)^4, \tag{11}$$

$$\frac{\partial}{\partial x} = \left(1 + \frac{\Delta x^2}{6}\delta_x^2\right)^{-1}\delta_{x0} + O(\Delta x)^4 \tag{12}$$

for the first derivatives from (8) and

$$\frac{\partial^2}{\partial x^2} = \left(1 + \frac{\Delta x^2}{12}\delta_x^2\right)^{-1}\delta_x^2 + O(\Delta x)^4 \tag{13}$$

for the second derivatives from (9).

Using the approximations (11) and (13), the first two equations of (1) are approximated as

$$u_t + \left(1 - \frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0}(u^2) + \left(1 - \frac{\Delta y^2}{6}\delta_y^2\right)\delta_{y0}(uv) + \left(1 - \frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0}p$$
$$= \frac{1}{R}\left(1 + \frac{\Delta x^2}{12}\delta_x^2\right)^{-1}\delta_x^2 u + \frac{1}{R}\left(1 + \frac{\Delta y^2}{12}\delta_y^2\right)^{-1}\delta_y^2 u + f_1 + O(\Delta)^4,$$

$$v_t + \left(1 - \frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0}(uv) + \left(1 - \frac{\Delta y^2}{6}\delta_y^2\right)\delta_{y0}(v^2) + \left(1 - \frac{\Delta y^2}{6}\delta_y^2\right)\delta_{y0}p$$
$$= \frac{1}{R}\left(1 + \frac{\Delta x^2}{12}\delta_x^2\right)^{-1}\delta_x^2 v + \frac{1}{R}\left(1 + \frac{\Delta y^2}{12}\delta_y^2\right)^{-1}\delta_y^2 v + f_2 + O(\Delta)^4.$$

We have used the symbol $O(\Delta)^4$ for $O(\Delta x)^4 + O(\Delta y)^4$. The discretization of the derivative in time is discussed in Section 7. Operating on these last equations with the product

$$\left(1 + \frac{\Delta x^2}{12}\delta_x^2\right)\left(1 + \frac{\Delta y^2}{12}\delta_y^2\right), \tag{14}$$

we obtain

$$\left(1 + \frac{\Delta x^2}{12}\delta_x^2\right)\left(1 + \frac{\Delta y^2}{12}\delta_y^2\right)\left[u_t + \left(1 - \frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0}(u^2) + \left(1 - \frac{\Delta y^2}{6}\delta_y^2\right)\delta_{y0}(uv) + \left(1 - \frac{\Delta x_2}{6}\delta_x^2\right)\delta_{x0}p\right]$$
$$= \frac{1}{R}\left(1 + \frac{\Delta y^2}{12}\delta_y^2\right)\delta_x^2 u + \frac{1}{R}\left(1 + \frac{\Delta x^2}{12}\delta_x^2\right)\delta_y^2 u + \left(1 + \frac{\Delta x^2}{12}\delta_x^2 + \frac{\Delta y^2}{12}\delta_y^2\right)f_1 + O(\Delta)^4$$

for the first component of the velocity and similarly for the other component. The stencil for the second-order difference terms from the Laplacian has the shape

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array}$$

where the scheme is centred about the centre of the stencil. The coefficients for the difference in the x-direction are

$$\frac{1}{12}\begin{bmatrix} 1 & -2 & 1 \\ 10 & -20 & 10 \\ 1 & -2 & 1 \end{bmatrix}.$$

The terms for the first difference in x for the convection terms and pressure gradient become

$$\left(1 + \frac{\Delta x^2}{12}\delta_x^2\right)\left(1 + \frac{\Delta y^2}{12}\delta_y^2\right)\left(1 - \frac{\Delta x^2}{6}\delta_x^2\right)\delta_{x0} + O(\Delta)^4 = \left(1 + \frac{\Delta y^2}{12}\delta_y^2 - \frac{\Delta x^2}{12}\delta_x^2\right)\delta_{x0} + O(\Delta)^4.$$

The stencil for the terms in this last expression (other than the $O(\Delta)^4$ terms) has the shape

$$\begin{array}{ccccc} & \bullet & & \bullet & \\ \bullet & \bullet & \times & \bullet & \bullet \\ & \bullet & & \bullet & \end{array}$$

where the $\times$ marks the centre point. The difference approximations in the y-direction are similar. Notice that the stencil must be modified for points one grid spacing from the boundary. The modifications are discussed in Section 5.

To insure the regularity, the pressure gradient expression is modified to be

$$\left(1+\frac{\Delta y^2}{12}\delta_y^2-\frac{\Delta x^2}{12}\delta_x^2\right)\delta_{x0}p+\gamma\Delta x^4\delta_x^4\delta_{x+}p.$$

As shown in Section 6, the term with γ positive will insure the regularity of the solution. Notice that this term does not degrade the accuracy of the difference formula.

Thus the scheme for the first momentum equation, with the exception of the time differencing is

$$\begin{aligned}\left(1+\frac{\Delta x^2}{12}\delta_x^2+\frac{\Delta y^2}{12}\delta_y^2\right)u_t+\left(1+\frac{\Delta y^2}{12}\delta_y^2-\frac{\Delta x^2}{12}\delta_x^2\right)\delta_{x0}(u^2)+\left(1-\frac{\Delta y^2}{12}\delta_y^2+\frac{\Delta x^2}{12}\delta_x^2\right)\delta_{y0}(uv)\\+\left(1+\frac{\Delta y^2}{12}\delta_y^2-\frac{\Delta x^2}{12}\delta_x^2\right)\delta_{x0}p+\gamma\Delta x^4\delta_x^4\delta_{x+}p\\=\frac{1}{R}\left(1+\frac{\Delta y^2}{12}\delta_y^2\right)\delta_x^2u+\frac{1}{R}\left(1+\frac{\Delta x^2}{12}\delta_x^2\right)\delta_y^2u+\left(1+\frac{\Delta x^2}{12}\delta_x^2+\frac{\Delta y^2}{12}\delta_y^2\right)f_1\end{aligned}\tag{15}$$

and similarly for the other momentum equation. Notice that the terms on the right-hand side of (15) are essentially the standard fourth-order-accurate scheme of the Poisson equation derived by Rosser.[12]

We next consider the approximation of the divergence equation (2) or (4). Using the approximation (12) on (2), we have

$$\left(1+\frac{\Delta x^2}{6}\delta_x^2\right)^{-1}\delta_{x0}u+\left(1+\frac{\Delta y^2}{6}\delta_y^2\right)^{-1}\delta_{y0}v=g+O(\Delta)^4.$$

Operating with the product

$$\left(1+\frac{\Delta x^2}{6}\delta_x^2\right)\left(1+\frac{\Delta y^2}{6}\delta_y^2\right),$$

we obtain

$$\left(1+\frac{\Delta y^2}{6}\delta_y^2\right)\delta_{x0}u+\left(1+\frac{\Delta x^2}{6}\delta_x^2\right)\delta_{y0}v=\left(1+\frac{\Delta x^2}{6}\delta_x^2+\frac{\Delta y^2}{6}\delta_y^2\right)g+O(\Delta)^4.\tag{16}$$

The stencil for the terms for the differencing in u has the shape

$$\begin{matrix}\bullet & & \bullet\\ \bullet & \times & \bullet\\ \bullet & & \bullet\end{matrix}$$

where the $\times$ marks the centre point. The stencil for the differencing of v is similar but rotated by a quarter of a turn.

To insure the regularity of the scheme, we modify (16) to give the scheme

$$\left(1+\frac{\Delta y^2}{6}\delta_y^2\right)\delta_{x0}u+\gamma\Delta x^4\delta_x^4\delta_{x-}u+\left(1+\frac{\Delta x^2}{6}\delta_x^2\right)\delta_{y0}v+\gamma\Delta y^4\delta_y^4\delta_{y-}v=\left(1+\frac{\Delta x^2}{6}\delta_x^2+\frac{\Delta y^2}{6}\delta_y^2\right)g.\tag{17}$$

Notice that the divergence operator and the gradient operator are not adjoints of each other.

4. THE SIXTH-ORDER SCHEME

The sixth-order-accurate scheme for the Navier–Stokes equations is based on expanding (8) to terms that are $O(\Delta x)^6$. The approximation for the first derivative is

$$\frac{\partial}{\partial x} = \left(1 - \frac{\Delta x^2}{6}\delta_x^2 + \frac{\Delta x^4}{30}\delta_x^4\right)\delta_{x0} + O(\Delta x)^6. \tag{18}$$

This equation is used to approximate first derivatives in the convection terms and the pressure gradient. It has a stencil involving seven points and for the divergence equation it is desirable to find a formula of sixth-order accuracy with a smaller stencil. An approximation giving a smaller stencil is

$$\frac{\partial}{\partial x} = \left(1 + \frac{\Delta x^2}{5}\delta_x^2\right)^{-1}\left(1 + \frac{\Delta x^2}{30}\delta_x^2\right)\delta_{x0} + O(\Delta x)^6. \tag{19}$$

Similarly we obtain from (9)

$$\frac{\partial^2}{\partial x^2} = \left(1 + \frac{2\Delta x^2}{15}\delta_x^2\right)^{-1}\left(1 + \frac{\Delta x^2}{20}\delta_x^2\right)\delta_x^2 + O(\Delta x)^6 \tag{20}$$

for the second derivatives.

Using the approximations (18) and (20), we have that the first of the two components of (1) may be approximated as

$$u_t + \left(1 - \frac{\Delta x^2}{6}\delta_x^2 + \frac{\Delta x^4}{30}\delta_x^4\right)\delta_{x0}(u^2) + \left(1 - \frac{\Delta y^2}{6}\delta_y^2 + \frac{\Delta y^4}{30}\delta_y^4\right)\delta_{y0}(uv) + \left(1 - \frac{\Delta x^2}{6}\delta_x^2 + \frac{\Delta x^4}{30}\delta_x^4\right)\delta_{x0}\, p$$

$$= \frac{1}{R}\left(1 + \frac{2\Delta x^2}{15}\delta_x^2\right)^{-1}\left(1 + \frac{\Delta x^2}{20}\delta_x^2\right)\delta_x^2 u + \frac{1}{R}\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)^{-1}\left(1 + \frac{\Delta y^2}{20}\delta_y^2\right)\delta_y^2 u + f_1 + O(\Delta)^6$$

and similarly for the second momentum equation.

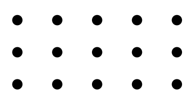
Operating through with the product

$$\left(1 + \frac{2\Delta x^2}{15}\delta_x^2\right)\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right), \tag{21}$$

we obtain

$$\begin{aligned}
&\left(1 + \frac{2\Delta x^2}{15}\delta_x^2\right)\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\Bigg[u_t + \left(1 - \frac{\Delta x^2}{6}\delta_x^2 + \frac{\Delta x^4}{30}\delta_x^4\right)\delta_{x0}(u^2) \\
&\quad + \left(1 - \frac{\Delta y^2}{6}\delta_y^2 + \frac{\Delta y^4}{30}\delta_y^4\right)\delta_{y0}(uv) + \left(1 - \frac{\Delta x^2}{6}\delta_x^2 + \frac{\Delta x^4}{30}\delta_x^4\right)\delta_{x0} p\Bigg] \\
&\quad = \frac{1}{R}\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\left(1 + \frac{\Delta x^2}{20}\delta_x^2\right)\delta_x^2 u + \frac{1}{R}\left(1 + \frac{2\Delta x^2}{15}\delta_x^2\right)\left(1 + \frac{\Delta y^2}{20}\delta_y^2\right)\delta_y^2 u \\
&\quad + \left(1 + \frac{2\Delta x^2}{15}\delta_x^2\right)\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right) f_1 + O(\Delta)^6.
\end{aligned}$$

The stencil for the second-order difference terms in the x-direction has the shape

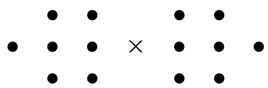


centred around the centre point. Notice that this difference approximation requires modification for two points away from the boundary.

The terms for the first-order difference in x become

$$\begin{aligned}&\left(1+\frac{2\Delta x^2}{15}\delta_x^2\right)\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{6}\delta_x^2+\frac{\Delta x^4}{30}\delta_x^4\right)\delta_{x0}+O(\Delta)^6\\ &=\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{30}\delta_x^2+\frac{\Delta x^4}{90}\delta_x^4\right)\delta_{x0}+O(\Delta)^6\\ &=\left[\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{30}\delta_x^2\right)+\frac{\Delta x^4}{90}\delta_4\right]\delta_{x0}+O(\Delta)^6.\end{aligned}$$

The stencil for these terms (other than the $O(\Delta)^6$ terms) has the shape



where the $\times$ marks the centre point.

To insure the regularity, we take the pressure gradient expression for the first momentum equation to be

$$\left[\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{30}\delta_x^2\right)+\frac{\Delta x^4}{90}\delta_x^4\right]\delta_{x0}p-\gamma\Delta x^6\delta_x^6\delta_{x+}p.$$

As shown in Section 6, the term with γ positive will insure the regularity of the solution and again the additional regularity term does not degrade the order of accuracy.

Thus the scheme, with the exception of the time differencing, is

$$\begin{aligned}&\left(1+\frac{2\Delta x^2}{15}\delta_x^2\right)\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)u_t+\left[\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{30}\delta_x^2\right)+\frac{\Delta x^4}{90}\delta_x^4\right]\delta_{x0}(u^2)\\ &\quad+\left[\left(1+\frac{2\Delta x^2}{15}\delta x^2\right)\left(1-\frac{\Delta y^2}{30}\delta_y^2\right)+\frac{\Delta y^4}{90}\delta_y^4\right]\delta_{y0}(uv)\\ &\quad+\left[\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{30}\delta_x^2\right)+\frac{\Delta x^4}{90}\delta_x^4\right]\delta_{x0}p-\gamma\Delta x^6\delta_x^6\delta_{x+}p\\ &=\frac{1}{R}\left[\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1+\frac{\Delta x^2}{20}\delta_x^2\right)\delta_x^2u+\left(1+\frac{2\Delta x^2}{15}\delta_x^2\right)\left(1+\frac{\Delta y^2}{20}\delta_y^2\right)\delta_y^2u\right]\\ &\quad+\left(1+\frac{2\Delta x^2}{15}\delta_x^2\right)\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)f_1\end{aligned}\tag{22}$$

and similarly for the other momentum equation.

We next consider the approximation of the divergence equation (2) or (4). Using the approximation (19) on (2), we have

$$\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)^{-1}\left(1+\frac{\Delta x^2}{30}\delta_x^2\right)\delta_{x0}u+\left(1+\frac{\Delta y^2}{5}\delta_y^2\right)^{-1}\left(1+\frac{\Delta y^2}{30}\delta_y^2\right)\delta_{y0}v=g+O(\Delta)^6.$$

Operating with the product

$$\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)\left(1+\frac{\Delta y^2}{5}\delta_y^2\right),$$

we obtain

$$\begin{aligned}\left(1+\frac{\Delta y^2}{5}\delta_y^2\right)\left(1+\frac{\Delta x^2}{30}\delta_x^2\right)\delta_{x0}u+\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)\left(1+\frac{\Delta y^2}{30}\delta_y^2\right)\delta_{y0}v\\=\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)\left(1+\frac{\Delta y^2}{5}\delta_y^2\right)g+O(\Delta)^6.\end{aligned}\tag{23}$$

To insure the regularity of the scheme, we modify (23) as

$$\begin{aligned}\left(1+\frac{\Delta y^2}{5}\delta_y^2\right)\left(1+\frac{\Delta x^2}{30}\delta_x^2\right)\delta_{x0}u-\gamma\Delta x^6\delta_x^6\delta_{x-}u+\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)\left(1+\frac{\Delta y^2}{30}\delta_y^2\right)\delta_{y0}v-\gamma\Delta y^6\delta_y^6\delta_{y-}v\\=\left(1+\frac{\Delta x^2}{5}\delta_x^2\right)\left(1+\frac{\Delta y^2}{5}\delta_y^2\right)g.\end{aligned}\tag{24}$$

The stencil for the difference operator on u, other than the term multiplied by γ , is

$$\begin{matrix}\bullet & \bullet & & \bullet & \bullet\\ \bullet & \bullet & \times & \bullet & \bullet\\ \bullet & \bullet & & \bullet & \bullet\end{matrix}$$

where the $\times$ marks the centre point.

One disadvantage of the sixth-order scheme over the fourth-order scheme is that because of the necessary boundary modifications, the stencil for the Laplacian is not symmetric for the sixth-order scheme. Some implications of this are described in Section 7.

5. BOUNDARIES AND EXTRAPOLATION OF PRESSURE

For the higher-order methods the stencils are so wide that some modification of the schemes is required at boundaries. Also, for all the schemes the pressure values on the boundary must be determined by extrapolation. The regularity terms, those multiplied by γ are removed whenever they conflict with boundaries.

We index the grid points by non-negative integers starting from zero. For a rectangle the grid points are indexed by (i, j) for $i = 0, 1, 2, \ldots, M$ and $j = 0, 1, 2, \ldots, N$ for some integers M and N. We consider only the boundary points with $i = 0$; the other boundaries are handled similarly.

We use the identity (10) to replace backward differences with forward differences. In particular we use

$$\delta_x^2=\delta_{x+}\delta_{x-}=(1-\Delta x\delta_{x+}+\Delta x^2\delta_{x+}-\Delta x^3\delta_{x+}^3)\delta_{x+}^2+O(\Delta x^4).\tag{25}$$

We also use the relation

$$\delta_{x0}=\tfrac{1}{2}(\delta_{x+}+\delta_{x-})=\frac{1}{2}\left(\delta_{x+}+\frac{\delta_{x+}}{1+\Delta x\delta_{x+}}\right)=\delta_{x+}-\tfrac{1}{2}\Delta x\delta_{x+}^2+O(\Delta x)^2.\tag{26}$$

For the fourth-order scheme the differences for the convection and gradient terms need modification near the boundary. For $i = 1$ the terms

$$\left(1 + \frac{\Delta y^2}{12}\delta_y^2 - \frac{\Delta x^2}{12}\delta_x^2\right)\delta_{x0}$$

are replaced by

$$\left(1 + \frac{\Delta y^2}{12}\delta_y^2\right)\delta_{x0} - \frac{\Delta x^2}{12}\delta_x^2(\delta_{x+} - \tfrac{1}{2}\Delta x\delta_{x+}^2).$$

For the sixth-order scheme differences for both the Laplacian and the convection terms need modification near the boundary. From the Laplacian the terms

$$\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\left(1 + \frac{\Delta x^2}{20}\delta_x^2\right)\delta_x^2 u_{i,j}$$

for $i = 1$ are replaced by

$$\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\left(1 + \frac{\Delta x^2}{20}(1 - \Delta x\delta_{x+} + \Delta x^2\delta_{x+}^2 - \Delta x^3\delta_{x+}^3)\delta_{x+}^2\right)\delta_x^2 u.$$

The expression

$$\left(1 + \frac{\Delta x^2}{20}(1 - \Delta x\delta_{x+} + \Delta x^2\delta_{x+}^2 - \Delta x^3\delta_{x+}^3)\delta_{x+}^2\right)\delta_x^2\phi$$

at grid point i is

$$\frac{24\phi_{i-1} - 62\phi_i + 72\phi_{i+1} - 69\phi_{i+2} + 56\phi_{i+3} - 28\phi_{i+4} + 8\phi_{i+5} - \phi_{i+6}}{20\Delta x^2}.$$

Similar modifications are made at the other boundaries.

For the pressure gradient and convection terms we also use (25). At $i = 1$ the expression

$$\left[\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\left(1 - \frac{\Delta x^2}{30}\delta_x^2\right) + \frac{\Delta x^4}{90}\delta_x^4\right]\delta_{x0}p \tag{27}$$

from the sixth-order scheme (22) is replaced by

$$\left[\left(1 + \frac{2\Delta y^2}{15}\delta_y^2\right)\left(1 - \frac{\Delta x^2}{30}(1 - \Delta x\delta_{x+} + \Delta x^2\delta_{x+}^2 - \Delta x^3\delta_{x+}^3)\delta_{x+}^2\right)\right]\delta_{x0}p + \frac{\Delta x^4}{90}(1 - \tfrac{3}{2}\Delta x\delta_{x+})\delta_{x+}^4\delta_{x-}p. \tag{28}$$

In this last expression the fourth-order divided difference was modified using the relations

$$\delta_x^4 = \delta_x^2\delta_{x+}^2(1 - x\delta_{x+}) + O(\Delta^2)$$

and the central difference was modified using (26).

The expression

$$\left(1 - \frac{\Delta x^2}{30}(1 - \Delta x\delta_{x+} + \Delta x^2\delta_{x+}^2)\delta_{x+}^2\right)\delta_{x0}p_1$$

in (28) expands to

$$\frac{-27p_0 - 9p_1 + 37p_2 + 4p_3 - 9p_4 + 5p_5 - p_6}{60\Delta x}$$

and

$$\frac{\Delta x^4}{90}\delta_x^2\delta_{x+}^3(1-\tfrac{3}{2}\Delta x\delta_{x+})p_{1,j}$$

expands to

$$\frac{-5p_{0,j}+28p_{1,j}-65p_{2,j}+80p_{3,j}-55p_{4,j}+20p_{5,j}-3p_{6,j}}{180\Delta x}.$$

At $i=2$ the expression (27) is replaced by

$$\left(1+\frac{2\Delta y^2}{15}\delta_y^2\right)\left(1-\frac{\Delta x^2}{30}\delta_x^2\right)\delta_{x0}p+\frac{\Delta x^4}{90}\delta_x^4(1-\tfrac{1}{2}\Delta x\delta_{x+})\delta_{x+}p, \tag{29}$$

where we used (26) on the central first-order difference.

The values of the pressure on the boundaries must be set by extrapolation from values in the interior. For a scheme of order r the extrapolation should have order $r+1$ to insure that the scheme is exact for polynomials of degree r. In this work the formula used to determine $p_{0,j}$ was

$$\delta_{x+}^{r+1}p_{0,j}=0. \tag{30}$$

For the fourth-order scheme the extrapolation is the fifth-order formula

$$p_{0,j}=5p_{1,j}-10p_{2,j}+10p_{3,j}-5p_{4,j}+p_{5,j}. \tag{31}$$

Similarly, for the sixth-order scheme the extrapolation is the seventh-order formula

$$p_{0,j}=7p_{1,j}-21p_{2,j}+35p_{3,j}-35p_{4,j}+21p_{5,j}-7p_{6,j}+p_{7,j}.$$

Values of the pressure must also be set in the corners of the Cartesian grids. In this work the formula used to determine $p_{0,0}$ was

$$\delta_{d+}^{r+1}p_{0,0}=0, \tag{32}$$

where d represents the diagonal direction. Thus in place of (31) at the corner with grid indices (0,0) we have

$$p_{0,0}=5p_{1,1}-10p_{2,2}+10p_{3,3}-5p_{4,4}+p_{5,5}.$$

The sixth-order extrapolation is similarly

$$p_{0,0}=7p_{1,1}-21p_{2,2}+35p_{3,3}-35p_{4,4}+21p_{5,5}-7p_{6,6}+p_{7,7}.$$

Other extrapolation formulae could be used in place of those given here. The main consideration is that the order of extrapolation be high enough not to affect the overall order of accuracy.

6. THE REGULARITY OF THE SCHEMES

In this section we check the regularity of the schemes. As shown in Reference 13, a scheme must be regular in order to insure that the solution is smooth. We consider only the steady equations, since the theory has only been developed for the steady state case. The importance of regularity is shown in the examples in Section 8.

To test the regularity of the fourth-order scheme consisting of (15), the similar formula for v and (17), we examine the symbol of the principal part of the scheme. We use the Fourier transform to determine the symbol by replacing $u_{l,m}$ by $\hat{u}e^{2i(x\theta_1+y\theta_2)}$ and similarly for v and p. Because of the factor 2 in the exponential, we are only concerned with θ_1 and θ_2 in the range $-\pi/2$ to $\pi/2$.

We may write the symbol as

$$\begin{pmatrix} L(\theta_1,\theta_2) & 0 & \mathrm{i}G(\theta_1,\theta_2)/\Delta x \\ 0 & L(\theta_1,\theta_2) & \mathrm{i}G(\theta_2,\theta_1)/\Delta y \\ \mathrm{i}D(\theta_1,\theta_2)/\Delta x & \mathrm{i}D(\theta_2,\theta_1)/\Delta y & 0 \end{pmatrix}, \tag{33}$$

where

$$L(\theta_2,\theta_2) = 4\left(1-\frac{\sin^2\theta_2}{3}\right)\frac{\sin^2\theta_1}{\Delta x^2} + 4\left(1-\frac{\sin^2\theta_1}{3}\right)\frac{\sin^2\theta_2}{\Delta y^2}, \tag{34}$$

$$G(\theta_1,\theta_2) = \left(1-\frac{\sin^2\theta_2}{3}+\frac{\sin^2\theta_1}{3}\right)\sin(2\theta_1) + 2^5\gamma\sin^5\theta_1\mathrm{e}^{\mathrm{i}\theta_1}, \tag{35}$$

$$D(\theta_1,\theta_2) = \left(1-\frac{2\sin^2\theta_2}{3}\right)\sin(2\theta_1) + 2^5\gamma\sin^5\theta_1\mathrm{e}^{\mathrm{i}\theta_1}. \tag{36}$$

The determinant of this system is

$$L(\theta 1,\theta_2)[G(\theta_1,\theta_2)D(\theta_1,\theta_2)/\Delta x^2 + G(\theta_2,\theta_1)D(\theta_2,\theta_1)/\Delta y^2]. \tag{37}$$

The scheme is regular precisely when this determinant vanishes only for $\theta_1=\theta_1=0$ with $|\theta_1|$ and $|\theta_2|$ less than or equal to $\pi/2$. The factor $L(\theta_1,\theta_2)$ so we need consider only the other factor in (17). We first evaluate the product $G(\theta_1,\theta_2)D(\theta_1,\theta_2)$. We have

$$\begin{aligned} G(\theta_1,\theta_2)D(\theta_1,\theta_2) = {} & [(1-\tfrac{1}{3}\sin^2\theta_2+\tfrac{1}{3}\sin^2\theta_1)(1-\tfrac{2}{3}\sin^2\theta_2) \\ & + \mathrm{i}2^5\gamma\sin^4\theta_1(1-\tfrac{1}{2}\sin^2\theta_2+\tfrac{1}{6}\sin^2\theta_1)]\sin^2(2\theta_1) \\ & + 2^{10}\gamma^2\sin^{10}\theta_1 - \mathrm{i}2^5\gamma\left(\frac{\sin^2\theta_1+\sin^2\theta_2}{3}\right)\sin^6\theta_1\sin(2\theta_1). \end{aligned}$$

An examination of this expression easily shows that the real part is a summation of non-negative terms and for $|\theta_1|$ and $|\theta_2|$ less than or equal to $\pi/2$ it vanishes only for $\theta_1=0$. Thus $G(\theta_1,\theta_2)D(\theta_1,\theta_2)/\Delta x^2 + G(\theta_2,\theta_1)D(\theta_2,\theta_1)/\Delta y^2$ does not vanish except for θ_1 and θ_2 both zero.

The regularity of the sixth-order scheme is analysed similarly. The symbol has the same form as (33) with

$$L(\theta_1,\theta_2) = 4\left(1-\frac{8\sin^2\theta_2}{15}\right)\left(1-\frac{2\sin^2\theta_1}{10}\right)\frac{\sin^2\theta_1}{\Delta x^2} + 4\left(1-\frac{8\sin^2\theta_1}{15}\right)\left(1-\frac{2\sin^2\theta_2}{10}\right)\frac{\sin^2\theta_2}{\Delta y^2}, \tag{38}$$

$$G(\theta_1,\theta_2) = \left[\left(1-\frac{8\sin^2\theta_2}{15}\right)\left(1+\frac{2\sin^2\theta_1}{15}\right)+\frac{8\sin^4\theta_1}{45}\right]\sin(2\theta_1) + 2^7\gamma\sin^7\theta_1\mathrm{e}^{\mathrm{i}\theta_1}, \tag{39}$$

$$D(\theta_1,\theta_2) = \left(1-\frac{4\sin^2\theta_2}{5}\right)\left(1-\frac{2\sin^2\theta_1}{15}\right)\sin(2\theta_1) + 2^7\gamma\sin^7\theta_1\mathrm{e}^{-\mathrm{i}\theta_1}. \tag{40}$$

The symbol L as defined in (38) vanishes only when $\theta_1=\theta_2=0$ with $|\theta_1|$ and $|\theta_2|$ less than or equal to $\pi/2$.

For the sixth-order scheme the product $G(\theta_1, \theta_2)D(\theta_1, \theta_2)$ is

$$\begin{aligned} G(\theta_1, \theta_2)D(\theta_1, \theta_2) = {} & [(1 - \tfrac{8}{15}\sin^2\theta_2)(1 - \tfrac{4}{5}\sin^2\theta_2)(1 - \tfrac{4}{225}\sin^4\theta_1) \\ & + \tfrac{8}{45}\sin^4\theta_1(1 - \tfrac{4}{5}\sin^2\theta_2)(1 - \tfrac{2}{15}\sin^2\theta_1) \\ & + 2^7\gamma\sin^6\theta_1(1 - \tfrac{2}{3}\sin^2\theta_2 + \tfrac{4}{225}\sin^2\theta_1\sin^2\theta_2 \\ & + \tfrac{4}{45}\sin^4\theta_1)]\sin^2(2\theta_1) + 2^{14}\gamma^2\sin^{14}\theta_1 \\ & - \frac{2^9 \mathrm{i}\gamma}{15}(\sin^2\theta_1(1 + \tfrac{2}{3}\sin^2\theta_1) + \sin^2\theta_2(1 - \tfrac{2}{3}\sin^2\theta_1))\sin^8\theta_1\sin(2\theta_1). \end{aligned}$$

As with the fourth-order scheme, with γ positive this expression vanished only for $\theta_1 = 0$. Therefore the expression (37) for the sixth-order scheme does not vanish for non-zero values of θ_1 and θ_2 and thus the scheme is regular.

7. THE TEMPORAL DIFFERENCING

In this section we discuss the temporal differencing of the time-dependent Navier–Stokes equations. We consider only schemes based on multistep methods from ordinary differential equations. To simplify the discussion, we consider the Navier–Stokes and Stokes equations in the form

$$\vec{u}_t = \mathscr{L}\vec{u} - \vec{\nabla}p + \vec{f}, \qquad \nabla \cdot \vec{u} = g, \tag{41}$$

where $\vec{u}$ denotes the velocity and p denotes the pressure. The operator $\mathscr{L}$ represents the terms involving spatial derivatives of the velocity. Our approach is motivated by the similarity of the system (41) to differential algebraic systems for ordinary differential equations. The incompressible Navier–Stokes equations are similar to differential algebraic systems of index two.[14]

For the system (41) the temporal differencing using a general multistep method is defined by

$$\frac{1}{\Delta t}\sum_{k=0}^{K} \alpha_k \vec{u}^{n-k} = \sum_{k=0}^{K} \beta_k \mathscr{L}\vec{u}^{n-k} - \sum_{k=0}^{K} \beta_k \vec{\nabla}p^{n-k} + \sum_{k=0}^{K} \beta_k \vec{f}^{n-k}, \qquad \nabla \cdot \vec{u}^n = g^n. \tag{42}$$

The two arrays of coefficients α_k and β_k are normalized by

$$\sum_{k=0}^{K} \alpha_k = 0, \qquad \sum_{k=0}^{K} \beta_k = 1.$$

Any of the second-order, fourth-order or sixth-order spatial differencing can be used with any of these time-dependent schemes. We consider the temporal differencing (42) to be applied before the spatial differencing, such as applying the operations (14) or (21).

The stability of the scheme depends on the two polynomials

$$\mathscr{A}(z) = \sum_{k=0}^{K} \alpha_k z^{K-k}, \qquad \mathscr{B}(z) = \sum_{k=0}^{K} \beta_k z^{K-k}.$$

A necessary condition for the stability of the overall scheme is that the polynomial $\mathscr{A}$ satisfy the standard root condition for stability in the sense of ordinary differential equations; that is, the roots of $\mathscr{A}(z) = 0$ must be inside the unit circle or simple on the unit circule.[15,16]

As the following theorem shows, for the standard multistep method (42), stability also requires that the polynomial $\mathscr{B}(z)$ satisfy the root condition.

Theorem 1

A necessary condition for stability of the multistep method (42) is that the roots of $\mathscr{B}(z) = 0$ be inside the unit circule or simple on the unit circle.

Proof. Let z be a root of $\mathscr{B}(z) = 0$ and let q be any non-constant function of the spatial variables. A solution of (42) is constructed by setting $\vec{u}^\nu = 0$ and $p^\nu = z^\nu q$. If the magnitude of z is larger than unity, then this solution will be unbounded in norm. If z is a multiple root with magnitude unity, then take $p^\nu = \nu z^\nu q$. Thus it is necessary for $\mathscr{B}(z)$ to satisfy the root condition. □

As is seen in the proof, the roots of the polynomial $\mathscr{B}(z)$ govern the growth of the pressure errors. The restriction on the roots of $\mathscr{B}(z)$ is a result of the pressure appearing only in the spatial differencing portion of the equations. Similar situations occur with semiexplicit differential algebraic equations of index two. However, defining $\bar{p}$ by

$$\bar{p}^n = \sum_{k=0}^{K} \beta_k p^{n-k},$$

we can replace (42) with

$$\frac{1}{\Delta t}\sum_{k=0}^{K} \alpha_k \vec{u}^{n-k} = \sum_{k=0}^{K} \beta_k \mathscr{L}\vec{u}^{n-k} - \vec{\nabla}\bar{p}^n - \vec{\nabla}\bar{p}^n + \sum_{k=0}^{K} \beta_k \vec{f}^{\,n-k}, \qquad \nabla\cdot\vec{u}_n = g^n. \tag{43}$$

The function $\bar{p}_n$ is at least a second-order accurate approximation to $p(t_n - \mu\Delta t)$, where

$$\mu = \sum_{k=0}^{K} k\beta_k.$$

Note that a disadvantage of this modified scheme is that the pressure is not obtained to the same accuracy as the velocity without some post-processing.

Theorem 1 is a severe limit on multistep schemes; however, the modified multistep scheme (43) allows for many schemes to be used. An adequate theory for the stability of schemes for the Navier–Stokes and Stokes equations has not been developed. Here we rely on the experience and theory of differential algebraic equations[14] to guide our choice.

Primarily we have used schemes based on backward time differences. These schemes, called backward differencing formula (BDF) schemes, are widely used for solving stiff ordinary differential equations and differential algebraic equations.[14] For these schemes $\beta_0 = 1$, the other $\beta_k = 0$ and the α_k are chosen from the formula

$$\frac{\partial}{\partial t} = \frac{\ln(1 - \Delta t\delta_{t-})}{\Delta t} = [1 + \tfrac{1}{2}\Delta t\delta_{t-} + \tfrac{1}{3}(\Delta t\delta_{t-})^2 + \tfrac{1}{4}(\Delta t\delta_{t-})^3 + \cdots]\delta_{t-},$$

truncated after K terms for a scheme with order of accuracy K. The coefficients α_k for a scheme of order K are given by

$$\alpha_0 = \sum_{j=1}^{K}\frac{1}{j}, \qquad \alpha_k = (-1)^k \sum_{j=k}^{K}\binom{j}{k}\frac{1}{j} \quad \text{for } k = 1, \cdots, K.$$

Our finite difference schemes are obtained by replacing the temporal derivative in equations (15) and (22) with the BDF operator of order K. In this paper we consider only $K = 2$, 3 and 4.

These schemes are stable as multistep schemes for ordinary differential equations for $K \leqslant 6$, but unstable for $K = 7$ and possibly all larger values of K.[17] In the numerical tests the BDF schemes outperformed the non-BDF schemes.

The backward differencing schemes we have used are as follows.

1. Second-order backward in time ($K = 2$): $\alpha_0 = \frac{3}{2}, \alpha_1 = -2, \alpha_2 = \frac{1}{2}; \beta_0 = 1$.
2. Third-order backward in time ($K = 3$): $\alpha_0 = \frac{11}{6}, \alpha_1 = -3, \alpha_2 = \frac{3}{2}, \alpha_3 = -\frac{1}{3}; \beta_0 = 1$.
3. Fourth-order backward in time ($K = 4$): $\alpha_0 = \frac{25}{12}, \alpha_1 = -4, \alpha_2 = 3, \alpha_3 = -\frac{4}{3}, \alpha_4 = \frac{1}{4}; \beta_0 = 1$.

In addition, three schemes were used in the first time steps, when the above schemes could not be used since they require several past time steps.

4. Crank–Nicolson (second-order-accurate): $\alpha_0 = 1, \alpha_1 = -1; \beta_0 = \beta_1 = \frac{1}{2}$.
5. Fourth-order in time using three time levels: $\alpha_0 = \frac{1}{2}, \alpha_1 = 0, \alpha_2 = -\frac{1}{2}; \beta_0 = \frac{1}{6}, \beta_1 = \frac{2}{3}, \beta_2 = \frac{1}{6}$.
6. Fourth-order in time using four time levels: $\alpha_0 = \frac{17}{24}, \alpha_1 = \alpha_2 = -\frac{3}{8}, \alpha_3 = \frac{1}{24}; \beta_0 = \frac{1}{4}, \beta_1 = \frac{3}{4}$.

These schemes are obtained by factoring the backward operators as done in Sections 3 and 4 for spatial difference operators. (Scheme 5 is equivalent to (12) applied in time rather than space.)

As multistep schemes, as in (42), schemes 5 and 6 are unstable by Theorem 1, but appear stable when used as in (43). Runs using the second-order scheme 1 used scheme 4 to compute the first time step. Runs using the third-order-accurate scheme 2 used scheme 4 for the first time step and scheme 1 for the second time step. Subsequent steps then used scheme 2.

Runs using the fourth-order scheme 3 used scheme 4 to compute the first time step, scheme 1 to compute the second time step and scheme 2 to compute the third time step. Subsequent steps then used scheme 3. Other choices for the initializing schemes could also be used. The use of a second-order-accurate scheme to initialize a fourth-order-accurate one appears not to reduce the overall order of accuracy. There was no reason to use a first-order-accurate scheme. The BDF schemes are dissipative of order two when applied to parabolic equations. For dissipative schemes with order of accuracy r, initializing schemes may be accurate of order $r - 2$ and still have the overall order of accuracy be r.

For the Stokes equations, because they are linear, the determination of the solution at the next time step requires the solution of a linear system. The system can be written as

$$\alpha_0 \frac{\vec{u}^n}{\Delta t} - \beta_0 \mathscr{L}\vec{u}^n + \vec{\nabla}\bar{p}^n = -\sum_{k=1}^{K} \alpha_k \frac{\vec{u}^{n-k}}{\Delta t} + \sum_{k=1}^{K} \beta_k \mathscr{L}\vec{u}^{n-k} + \sum_{k=0}^{K} \beta_k \vec{f}^{n-k}, \qquad \nabla \cdot \vec{u}^n = g_n. \tag{44}$$

This system determines the solution $\vec{u}^n$, and $\bar{p}^n$ for the new time level. Note that the spatial operators, such as (14) or (21) must be applied to (44).

We have used a preconditioned GMRES method[10] for the solution of this linear system. The usual method was GMRES(7) with a restart. The preconditioner was an inversion of the operator.

$$\alpha_0 \frac{1}{\Delta t} - \beta_0 \nabla^2 \tag{45}$$

on the first two equations in the system. Other methods of solving the linear system could also be used. An advantage of GMRES is that it does not require the system to be symmetric.

As mentioned, the preconditioner for the system (44) was the inversion of the operator (45); for the second-order- and fourth-order-accurate schemes this was done using the preconditioned conjugate gradient method with SSOR as the preconditioner. For the sixth-order scheme, because the operator is not symmetric, the GMRES method was used. Although accurate solutions were obtained, this method was not particularly efficient. More experience is needed with these methods to improve the overall efficiency.

Because the pressure can only be determined to within an additive constant, the system (44) is singular. Moreover, the existence of the solution is dependent on satisfying the integrability condition. An important issue is the choice of norms to determine convergence of the system (44). By requiring only that the quantity $\vec{\nabla} \cdot \vec{u}^n - g^n$ be constant as described in Reference 8, we effectively have a non-singular system. The average value of $\vec{\nabla} \cdot \vec{u}^n - g^n$ over the grid is a measure of the consistency of the data. In all the results shown in Section 8 this value is less than the errors themselves by several orders of magnitude.

For the non-linear Navier–Stokes equations we modify the equations to obtain a linear system for the solution that does not degrade the accuracy. We linearize the quadratic expressions u^2, uv and v^2 at time level n using the following idea. Consider two functions $A(t)$ and $B(t)$ depending on the independent variable t. Using the relations

$$\delta^r_{t-} A(t) = 0, \qquad \delta^r_{t-} B(t) = 0,$$

we obtain approximations $\bar{A}$ and $\bar{B}$ to order r for $A(t)$ and $B(t)$. The formulate for $\bar{A}^n$ is

$$\bar{A}^n = \sum_{k=1}^{r} (-1)^{k-1} \binom{r}{k} A^{n-k}$$

and similarly for $\bar{B}^n$.

The product AB involving past values of the variable t is approximated using the relation

$$(A - \bar{A})(B - \bar{B}) = O(\Delta t)^{2r}.$$

In particular, at $t_n = n\Delta t$ with $A^n = A(t_n)$ we can write

$$A^n B^n = A^n \bar{B}^n + \bar{A}^n B^n - \bar{A}^n \bar{B}^n + O(\Delta t)^{2r},$$

where $\bar{A}^n$ and $\bar{B}^n$ depend on values of t less than t_n.

We use these formulae for $t = n\Delta t$ and with A and B being velocity components. We take r equal to the number of time levels available in the scheme. The expressions $\partial(uv)/\partial x$ at time step n is approximated by

$$\partial(u^n \bar{v}^n + \bar{u}^n v^n - \bar{u}^n \bar{v}^n)/\partial x$$

and the spatial derivative is approximated using either the fourth-order or sixth-order method given previously. In this way the equation being solved for the solution at each time step is a linear system and the accuracy of the solution is not affected. For schemes 4–6 the above approximations were used only at the time step being solved for.

8. NUMERICAL EXPERIMENTS

Several tests are described in this section that illustrate the accuracy of the methods. The first set of tests checks the formal order of accuracy and the second set of tests examines the accuracy for an analytically known solution. The finite difference methods were implemented using the C programming language with double-precision variables.

The finite difference schemes were tested extensively to assure that when the velocity and pressure were polynomials of appropriate degree, the solutions satisfied the schemes to within machine precision. This served as both a test of the methods and a means to detect programming errors in the implementation of the methods. For example, the solution

$$\begin{aligned} u &= x^6 - y^6, & f_1 &= -30(x^4 - y^4) + 4x^3y^2, \\ v &= x^3y^3, & f_2 &= -6(xy^3 + x^3y) + 2x^4y, \\ p &= x^4y^2, & g &= 6x^5 + 3x^3y^2 \end{aligned}$$

was used with the steady state Stokes equations (3) and (4) for the sixth-order method. Other sixth-degree polynomials were used also. By considering these solutions after translations and rotations in the plane, a large class of solutions could be obtained. Similar tests were made for the fourth-order scheme.

For positive Reynolds numbers, because of the quadratic convection terms, the fourth-order scheme is exact for all polynomials of degree two for the velocity and of degree four for the pressure. Similarly, the sixth-order method is exact for third-degree polynomials for the velocity and sixth-degree polynomials for the pressure. In all these tests with polynomial solutions the solutions were computed to within machine precision. Similar tests were used to check the temporal differencing.

For a test of the method on less trivial application we take the solution used by Pearson,[18] Chorin[19] and others to test their methods. The solution is given by

$$u = -e^{-2t/R}\cos x \sin y, \qquad v = e^{-2t/R}\sin x \cos y, \qquad p = -\tfrac{1}{4}e^{-4t/R}[\cos(2x) + \sin(2y)]$$

on the square $0 < x < \pi, 0 < y < \pi$.

We used Reynolds numbers of 2 and 100. For Reynolds number 2 the solution changes quite rapidly and this tests the efficiency of the time integration. The solutions were computed up to time 0·5. For Reynolds number 100 this solution decays quite slowly in time and so the errors are principally due to the spatial discretization.

For more realistic flows the solutions are difficult to compute at high Reynolds number, because it is difficult to solve the linear systems. Improvements in the preconditioning methods will no doubt improve the efficiency of the numerical computations. Work on preconditioners for these systems will be reported in Reference 20. These schemes have been incorporated in a code employing domain decomposition to compute solutions in regions with complex geometry. A description of this code with computational results will be published separately.

Computational results showing the order of accuracy are displayed in Table I. In all tables the order of accuracy of the method is given as an ordered pair (p, q), where p is the temporal order of accuracy and q is the spatial order of accuracy. The order of accuracy r is computed via the formula

$$r = \frac{\ln[\mathrm{err}(h_1)/\mathrm{err}(h_2)]}{\ln(h_1/h_2)},$$

using two successive values of the grid spacing h_1 and h_2 and the corresponding errors $\mathrm{err}(h_1)$ and $\mathrm{err}(h_2)$.

The first three cases in Table I demonstrate the accuracy of the (3, 4) scheme. The time steps are chosen so that the overall convergence rate should be $O(h^4)$. (The computations proceeded to the first

Table I. Order of accuracy of computed solutions, $R = 2$

Case	Order	Δt	n	Error u	Order u	Error p	Order p
1	(3,4)	$(\frac{1}{10})^{4/3}$	10	8·478(−5)		1·724(−3)	
2	(3,4)	$(\frac{1}{20})^{4/3}$	20	1·668(−6)	5·7	4·041(−5)	5·4
3	(3,4)	$(\frac{1}{40})^{4/3}$	40	3·142(−8)	5·7	1·829(−6)	4·5
4	(4,4)	$\frac{1}{10}$	10	4·778(−5)		1·617(−3)	
5	(4,4)	$\frac{1}{20}$	20	2·102(−6)	4·5	3·379(−5)	5·6
6	(4,4)	$\frac{1}{30}$	30	2·493(−7)	5·3	6·298(−6)	4·1
7	(4,4)	$\frac{1}{40}$	40	1·339(−7)	2·2	6·067(−7)	8·1
8	(4,6)	$(\frac{1}{10})^{3/2}$	10	3·450(−5)		2·717(−4)	
9	(4,6)	$(\frac{1}{15})^{3/2}$	15	3·955(−6)	5.3	1·891(−5)	6·6
10	(4,6)	$(\frac{1}{20})^{3/2}$	20	7·445(−7)	5·8	3·444(−6)	5·9
11	(4,6)	$(\frac{1}{30})^{3/2}$	30	6·946(−8)	5·8	3·773(−7)	5·4

time larger than 0·5 with the given time step.) The errors show that for this series of runs the solution error is no worse than $O(h^4)$. The pressure errors were computed using the 'standard deviation' as a norm; that is, the norm is the mean square of the error minus the average error.[8]

Similarly, the order of accuracy of the four (4, 4) cases shown in Table I indicate that the scheme is fourth-order-accurate. The velocity for the case $h = \frac{1}{30}$ is fortuitously more accurate than the $O(h^4)$ would predict. This makes the computed order of accuracy between $h = \frac{1}{30}$ and $\frac{1}{40}$ lower than 4·0. The order of accuracy for the velocity computed for the cases $h = \frac{1}{20}$ and $\frac{1}{40}$ is 4·0. Similarly, the pressure error for $h = \frac{1}{40}$ is fortuitously low, giving a higher-than-expected value for r.

The last four cases in Table I show that the (4, 6) scheme is indeed sixth-order-accurate when the time step is chosen as $\Delta t = h^{3/2}$, where h is the grid spacing.

In Table II are shown the results of several runs with Reynolds number 100. Since in this case the decay rate in time is quite slow, the tests do not severely test the time integration. Cases 1–4 are for solutions computed for time up to 1 and cases 5–8 are for solutions computed for time up to 10. Notice that the error does not grow significantly between $t = 1$ and 10 and the computed orders of accuracy are similar to those in Table I.

Table III illustrates several features of the schemes using a Reynolds number of 2. The runs marked with an asterisk are those which used exact conditions for the initialization and the runs marked with a dagger are those for which the parameter γ was non-zero. Cases 1–4 show the effect of the use of the Crank–Nicolson scheme to compute the first time step for the second-order BDF scheme. Cases 1 and 3 used the exact solution to obtain the solution at $t = \Delta t$, while cases 2 and 4 used the Crank–Nicolson scheme to compute this first time step.

Similarly, for case 5 the exact solution was used at times Δt, $2\Delta t$ and $3\Delta t$ before using the fourth-order BDF method for the subsequent time steps. For case 6 the Crank–Nicolson method was used for the first time step, the second-order BDF for the second time step and the third-order BDF for the third time step as described in Section 7. A comparison of cases 5 and 6 shows that there is not a substantial difference in the errors introduced by using lower-order methods for initializing the computation. As shown in Table I, where the initialization is done as described in Section 7, the accuracy of the (4, 6) methods is not adversely affected by the initialization. For a discussion of the effect of the accuracy of initializing schemes on the overall accuracy see Reference 11, Section 10.6.

Cases 7 and 8 illustrate the effect of the regularizing parameter γ on the accuracy of the solution: $\gamma = 0$ for case 7 and $\gamma = 0{\cdot}001$ for case 8. The pressure error for case 7 is very oscillatory. The small value of γ in case 8 decreases the pressure error by a factor of three.

Similarly, for case 9 $\gamma = 0$, for case 10 it is 0·01 and for case 11 it is 0·02. These small values have a significant effect on the error, especially for the pressure.

Table II. Order of accuracy of computed solutions, $R = 100$

Case	Order	Δt	n	Error u	Order u	Error p	Order p
1	(4,4)	$\frac{1}{10}$	10	1·347(−3)		4·391(−3)	
2	(4,4)	$\frac{1}{20}$	20	6·521(−5)	4·4	1·068(−4)	5·4
3	(4,4)	$\frac{1}{30}$	30	8·974(−6)	4·9	1·129(−5)	5·5
4	(4,4)	$\frac{1}{40}$	40	2·087(−6)	5·2	2·320(−6)	5·5
5	(4,4)	$\frac{1}{10}$	10	2·695(−3)		5·449(−3)	
6	(4,4)	$\frac{1}{20}$	20	5·312(−5)	5·7	7·743(−5)	6·1
7	(4,4)	$\frac{1}{30}$	30	7·305(−6)	4·9	8·481(−6)	5·5
8	(4,4)	$\frac{1}{40}$	40	1·691(−6)	5·1	1·769(−6)	5·4

Table III. Effect of initial conditions and regularity

Case	Order	Δt	n	Error u	Error v	Error p
1*	(2.4)	0·01	20	1·984(−6)	1·770(−6)	4·058(−5)
2	(2,4)	0·01	20	1·982(−6)	1·769(−6)	4·058(−5)
3*	(2,4)	0·02	60	5·565(−6)	5·608(−6)	8·608(−5)
4	(2,4)	0·02	60	5·581(−6)	5·542(−6)	8·610(−5)
5*	(4,6)	0·05	20	7·327(−7)	7·253(−7)	3·986(−6)
6	(4,6)	0·05	20	1·337(−6)	1·271(−6)	4·856(−6)
7	(4,4)	0·02	40	7·469(−8)	5·753(−8)	1·745(−6)
8†	(4,4)	0·02	40	7·439(−8)	5·682(−8)	5·624(−7)
9	(4,6)	0·01	40	3·489(−8)	3·064(−8)	1·043(−5)
10†	(4,6)	0·01	40	1·317(−8)	1·278(−8)	6·162(−6)
11†	(4,6)	0·01	40	1·358(−8)	1·286(−8)	6·873(−7)

As discussed in Section 7, the method GMRES(7) was used to solve for the solution variables at each time step. An average of about 10 applications of GMRES(7) were required per time step.

9. CONCLUSIONS

In this paper we have presented several new finite difference schemes for the incompressible Navier–Stokes equations. We have shown that these schemes have a high order of accuracy.

The schemes are based on two spatial differencing methods, one a fourth-order-accurate method and one a sixth-order-accurate method. The temporal differencing methods are BDF methods of orders two, three and four. These temporal schemes can be used with either of the spatial differencing methods. The schemes as presented are for orthogonal Cartesian grids. The schemes can be used for both the steady state and time-dependent equations.

For the time-dependent methods the storage requirements of the second-order scheme appear to be greater than needed by fractional step methods. However, this is compensated for by the larger time steps and smaller stencils for the higher-order spatial discretization. For the higher-order accuracy in time the storage requirements are reasonable, increasing by one level of storage with each order of accuracy.

The methods presented in this paper have been demonstrated to be accurate and effective methods for solving the time-dependent incompressible Navier–Stokes and Stokes equations. Further research is needed to improve the efficiency of the solution of the linear systems, especially for higher-Reynolds-number solutions.

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